Calculus Worksheet: Limits of Functions (2) – Squeeze Theorem

1. f and g are functions such that $f(x) = \sqrt{x^2 + 2|x|}$ and $a \le g(x) \le b$, where a and b are real numbers.

Find

$$\lim_{x\to 0} f(x)g(x)$$

Note that $\lim\limits_{x\to 0} f(x) = 0$

Also $f(x) \cdot a \le f(x) \cdot g(x) \le f(x) \cdot b$

$\lim\limits_{x\to 0} f(x) \cdot a = 0 \cdot a = 0$, $\lim\limits_{x\to 0} f(x) \cdot b = 0 \cdot b = 0$

by the squeeze theorem $\lim\limits_{x\to 0} g(x) \cdot f(x) = 0.$

2. Find the following limits

a) $\lim_{x\to 0} x^3 \sin(\frac{1}{x})$

$-1 \le \sin\left(\frac{1}{x}\right) \le 1.$

multiply all terms of the inequality by x^3.

a) $x < 0$, $x^3 < 0$

$-x^3 \ge x^3 \sin\left(\frac{1}{x}\right) \ge +x^3$

$\lim\limits_{x\to 0^-} (-x^3) = 0$, $\lim\limits_{x\to 0^-} (x^3) = 0$

by the squeeze theorem $\lim\limits_{x\to 0^-} x^3 \sin\left(\frac{1}{x}\right) = 0$

b) $x > 0$, $x^3 > 0$

$-x^3 \le x^3 \sin\left(\frac{1}{x}\right) \le x^3$

$\lim\limits_{x\to 0^+} (-x^3) = 0$, $\lim\limits_{x\to 0^+} (x^3) = 0$

by the squeeze theorem $\lim\limits_{x\to 0^+} x^3 \sin\left(\frac{1}{x}\right) = 0$

Hence $\lim\limits_{x\to 0} x^3 \sin\left(\frac{1}{x}\right) = 0.$

b) $\lim_{x\to\infty} \frac{\sin^2(x)}{1-x^2}$

$0 \leq \sin^2(x) \leq 1$

as $x \to +\infty$, $1-x^2 < 0$

multiply all term of the inequality by $\frac{1}{1-x^2}$

$\frac{0}{1-x^2} \geq \frac{\sin^2(x)}{1-x^2} \geq \frac{1}{1-x^2} \Rightarrow 0 \geq \frac{\sin^2(x)}{1-x^2} \geq \frac{1}{1-x^2}$

as $x \to \infty$ $0 \to 0$ and $\frac{1}{1-x^2} \to 0$

by the squeeze theorem.

$$\lim_{x \to +\infty} \frac{\sin^2(x)}{1-x^2} = 0$$

c) $\lim_{x\to\infty} \frac{4x^2 - \sin(2x)}{3x^2+10}$

$-1 \leq \sin(2x) \leq 1$ range of $\sin(2x)$.

$\Downarrow$

$4x^2 - 1 \leq \cancel{\sin(2x)}\ 4x^2 - \sin(2x) \leq 4x^2+1$

Divide all terms by $3x^2+10$

$$\frac{4x^2-1}{3x^2+10} \leq \frac{4x^2-\sin(2x)}{3x^2+10} \leq \frac{4x^2+1}{3x^2+10}$$

$$\lim_{x\to+\infty} \frac{4x^2-1}{3x^2+10} = \frac{4}{3}, \quad \lim_{x\to\infty} \frac{4x^2+1}{3x^2+10} = \frac{4}{3}$$

by the squeeze theorem

$$\lim_{x\to+\infty} \frac{4x^2-\sin(2x)}{3x^2+10} = \frac{4}{3}$$