Calculus Worksheet: Differentiation of Inverse Functions (1)

review

If f^{-1} is the inverse of function f then

$$f(f^{-1}(x)) = x$$

If we let $u = f^{-1}(x)$

Then we have $f(u) = x$.

Differentiate both side of $f(u) = x$ to obtain

$$\frac{df}{du}\frac{du}{dx} = 1$$ (The chain rule has been used for the term $f(u)$)

The above may be written as

$$\frac{du}{dx} = \frac{1}{\frac{df}{du}}$$

Since $u = f^{-1}(x)$, the above may be written as

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'(u)} = \frac{1}{f'(f^{-1}(x))}$$

And finally

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))}$$

Where f' is the first derivative of f and $f'(f^{-1}(x))$ is the first derivative evaluated at $f^{-1}(x)$.

See example on how to use the above formula in next page.

Example 1: Use the above formula to find the first derivative of the inverse of the sine function written as

$$y = \sin^{-1}(x) \; , -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Let $f(x) = \sin(x)$ and $f^{-1}(x) = \sin^{-1}(x)$ and use the formula to write

$$\frac{dy}{dx} = \frac{d\sin^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))}$$

f' is the first derivative of f and is given by

$$f'(x) = \cos(x)$$

Hence

$$\frac{d\sin^{-1}(x)}{dx} = \frac{1}{\cos(f^{-1}(x))}$$

Let us now deal with the term $\cos(f^{-1}(x))$ which is equal to $\cos(\sin^{-1}(x))$

Let $\alpha = \sin^{-1}(x)$ and write

$$\cos(\sin^{-1}(x)) = \cos(\alpha) = \sqrt{1 - \sin^2(\alpha)}$$

Use the fact that $\sin(\sin^{-1}(x)) = x$ to write that

$$\sin^2(\alpha) = [\ \sin(\sin^{-1}(x))\ ]^2 = x^2$$

Substitute $\sin^2(\alpha)$ by x^2 above to obtain

$$\cos(f^{-1}(x)) = \cos(\sin^{-1}(x)) = \sqrt{1 - x^2}$$

Finally

$$\frac{d\sin^{-1}(x)}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

More examples are in this website.
www.analyzemath.com/calculus/Differentiation/inverse_trigonometric.html